EUROPEAN PUT OPTION PRICING MODEL WITH GRAM-CHARLIER EXPANSION IN THIRD MOMENTS

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ABSTRACT

The Black-Scholes model is one of the most popular and widely applied option pricing models in both academic and practical contexts developed by Black and Scholes (1973). The practical assumption in the Black-Scholes model is stock return following the normal distribution with constant volatility. However, many stock returns are not normally distributed, so should consider the skewness and kurtosis of the stock return. This developmental model adapts the Gram-Charlier expansion to adapt skewness and kurtosis to the Black-Scholes formula. Approximation method used is an alternative approach with Hermite polynomial. The observed stocks are SPG, C, and AXP by taking stock price data from November 11, 2016 to November 11, 2017 with maturity date at January 18, 2019 and interest rate (r) of 1.25%. After comparing the average MSE of both models, found that the third moment Gram-Charlier expansion is better than the Black-Scholes model in modeling SPG, C, and TSLA stock prices.

Keywords: Stock Option, Black-scholes, Gram-Charlier Expansion, Hermite Polynomial

INTRODUCTION

Option is a contract or agreement between two parties, where the first party as a buyer has the right to buy or sell from a second party that is the seller of a particular asset at a specified price and time. The Black-Scholes model is one of the popular option pricing models developed by Black and Scholes (1973). The Black-Scholes model assumes that stock return follows the normal distribution (skewness 0 and kurtosis 3) with constant volatility. Hull (1993) and Nattenburg (1994) point out that stock returns exhibit nonnormal skewness and kurtosis and that volatility skews are a consequence of empirical violations of the normality assumption.

Corrado and Su (1996) develop a method to incorporate option price adjustments for nonnormal skewness and kurtosis in an expanded Black–Scholes option pricing formula. Their method adapts a Gram–Charlier series expansion of the standard normal density function to yield an option price formula that is the sum of a Black–Scholes option price plus adjustment terms for nonnormal skewness and kurtosis.

In this paper, it will be explained that the pricing of options with returns not normally distributed can be found by considering the skewness ($\mu_3$) of the return. This model adapts the Gram-Charlier expansion to provide skewness and kurtosis adjustments to the Black-Scholes formula. Approximation method used is an alternative approach with Hermite polynomial.

Black-Scholes Model

The formula of the price of put option Black-Scholes model as present value of expected benefit of European put option.

$$P_{BS} = e^{-rT} E[\max(K - S_T, 0)]$$

$$= Ke^{-rT} N(-d_2) - S_0 N(-d_1).$$
**Hermite Polynomials**

Given density function of standard normal distribution \( n(z) D = \frac{d}{dz} \) as differentiation operator. Hermite Polynomials are a classical orthogonal polynomial sequence at interval \((-\infty, \infty)\), can be defined from Rodrigues Formula as follows:

\[
H_n(z)n(z) = (-D)^n n(z) \\
= (-1)^n \frac{d^n}{dz^n} n(z).
\]

(2)

Then we get,

\[
H_n(z) = (-1)^n e^{\frac{z^2}{2}} \frac{d^n}{dz^n} \left(e^{-\frac{z^2}{2}}\right).
\]

For \( n = 3 \), obtained :

\[
H_3(z)n(z) = (-D)^3 n(z) = (z^3 - 3z)n(z) \quad \rightarrow \quad H_3(z) = (z^3 - 3z)
\]

The properties of orthogonal polynomial Hermite i.e:

\[
\int_{-\infty}^{\infty} H_m(z) H_n(z) n(z) \, dz = \begin{cases} 0 & \text{jika } m \neq n \\ m! & \text{jika } m = n \end{cases}
\]

(3)

**Gram-Charlier Expansion**

Density function of Gram-Charlier expansion as follows:

\[
g(z) = \sum_{n=0}^{\infty} c_n H_n(z) n(z)
\]

(4)

Where \( n(z) \) is density function of standard distribution normal, \( H_n(z) \) nth order of polynomial Hermite. Coefficient \( c_n \) on equation (4) come from Polynomial Hermite. If equation (4) to side multiplied by \( H_m(z) \) then, integrated from the boundary \(-\infty\) until \( \infty \) then obtained:

\[
\int_{-\infty}^{\infty} g(z) H_m(z) \, dz = c_0 \int_{-\infty}^{\infty} H_0(z) H_m(z) n(z) \, dz + c_1 \int_{-\infty}^{\infty} H_1(z) H_m(z) n(z) \, dz + \cdots + c_m \int_{-\infty}^{\infty} H_m(z) H_m(z) n(z) \, dz + \cdots
\]

Using the properties of orthogonal polynomial Hermite in equation (3) obtained:

\[
\int_{-\infty}^{\infty} g(z) H_m(z) \, dz = 0 + 0 + \cdots + c_m \int_{-\infty}^{\infty} H_m(z) H_m(z) n(z) \, dz + \cdots
\]

\[
= c_m \int_{-\infty}^{\infty} H_m(z) H_m(z) n(z) \, dz
\]

\[
= c_m m!
\]

We get: 

\[
c_m = \frac{1}{m!} \int_{-\infty}^{\infty} g(z) H_m(z) \, dz
\]

From the above equation obtained the value of \( c_m \) as follows:

\[
c_0 = \frac{1}{0!} \int_{-\infty}^{\infty} g(z) H_0(z) \, dz = \int_{-\infty}^{\infty} g(z) \, dz = 1
\]

\[
c_1 = \frac{1}{1!} \int_{-\infty}^{\infty} g(z) H_1(z) \, dz = \int_{-\infty}^{\infty} g(z) z \, dz = E(z) = \mu_1
\]

\[
c_2 = \frac{1}{2!} \int_{-\infty}^{\infty} g(z) H_2(z) \, dz = \frac{1}{2!} \int_{-\infty}^{\infty} g(z) (z^2 - 1) \, dz = \frac{1}{2!} [\mu_2 - 1]
\]

\[
c_3 = \frac{1}{3!} \int_{-\infty}^{\infty} g(z) H_3(z) \, dz = \frac{1}{3!} \int_{-\infty}^{\infty} g(z) (z^3 - 3z) \, dz = \frac{1}{3!} [\mu_3 - 3\mu_1]
\]

... 

**Gram-Charlier** expansion is obtained as follows:

\[
g(z) = \sum_{n=0}^{\infty} c_n H_n(z) n(z) = n(z) \sum_{n=0}^{\infty} c_n H_n(z)
\]

\[
= n(z) [c_0 H_0(z) + c_1 H_1(z) + c_2 H_2(z) + \cdots]
\]
\[ g(z) = n(z)[H_0(z) + \mu_1 H_1(z) + \frac{1}{2!}[\mu_2 - 1]H_2(z) + \frac{1}{3!}[\mu_3 - 3\mu_1]H_3(z) + \cdots \]

Since the moment used only until the third moment then

\[ g(z) = n(z)[H_0(z) + \mu_1 H_1(z) + \frac{1}{2!}[\mu_2 - 1]H_2(z) + \frac{1}{3!}[\mu_3 - 3\mu_1]H_3(z) \]

In addition, given that \( z \) is standard normal distribution, then \( E(z) = \mu_i = 0 \) and \( E(z^2) = \mu_2 = 1 \) so that \textit{Gram-Charlier} expansion above become:

\[ g(z) = n(z)[1 + \frac{\mu_3}{3!}H_3(z)] \]

**European Put Option Price by Gram-Charlier Expansion**

With substituting \( S_T = e^{\sigma \sqrt{T}z} \) and \(-d_2 = \frac{\ln K - m}{\sigma \sqrt{T}}\) then

\[
P_3 = e^{-rT} \int_0^K (K - S_T)g(S_T)d(S_T)
= e^{-rT} \int_{\frac{\ln K - m}{\sigma \sqrt{T}}}^{\frac{\ln K - m}{\sigma \sqrt{T}}} (K - e^{z\sigma \sqrt{T} + m})n(z)\left(1 + \frac{\mu_3}{3!}H_3(z)\right)dz
= e^{-rT} \int_{\frac{\ln K - m}{\sigma \sqrt{T}}}^{\frac{\ln K - m}{\sigma \sqrt{T}}} (K - e^{z\sigma \sqrt{T} + m})n(z)dz + e^{-rT} \int_{-\infty}^{\frac{\ln K - m}{\sigma \sqrt{T}}} (K - e^{z\sigma \sqrt{T} + m})n(z) dz
= P_{BS} + \frac{\mu_3}{3!} e^{-rT} \int_{-\infty}^{\frac{\ln K - m}{\sigma \sqrt{T}}} (K - e^{z\sigma \sqrt{T} + m})n(z)H_3(z)dz
\]

let \( l_3 = e^{-rT} \int_{-\infty}^{\frac{\ln K - m}{\sigma \sqrt{T}}} (K - e^{z\sigma \sqrt{T} + m})n(z)H_3(z)dz \)

\[
l_3 = e^{-rT} \int_{-\infty}^{\frac{\ln K - m}{\sigma \sqrt{T}}} (K - e^{z\sigma \sqrt{T} + m})n(z)H_3(z)dz
= -e^{-rT} \int_{-\infty}^{\frac{\ln K - m}{\sigma \sqrt{T}}} (K - e^{z\sigma \sqrt{T} + m}) \frac{d}{dz} \left(\frac{d^2 n(z)}{dz^2}\right) dz
\]

let: \( u = K - e^{z\sigma \sqrt{T} + m}, du = -\sigma T e^{z\sigma \sqrt{T} + m}dz, dv = \frac{d}{dz} \left(\frac{d^2 n(z)}{dz^2}\right) dz, v = \left(\frac{d^2 n(z)}{dz^2}\right) \)

\[
l_3 = -e^{-rT} \int_{-\infty}^{\frac{ln K - m}{\sigma \sqrt{T}}} \left(\sigma \sqrt{T} e^{z\sigma \sqrt{T} + m}\right) \frac{d^2 n(z)}{dz^2} dz
= -e^{-rT} \int_{-\infty}^{\frac{ln K - m}{\sigma \sqrt{T}}} \sigma \sqrt{T} e^{z\sigma \sqrt{T} + m} \frac{d^2 n(z)}{dz^2} dz
\]

let: \( u = \sigma \sqrt{T} e^{z\sigma \sqrt{T} + m}, du = \sigma^2 T e^{z\sigma \sqrt{T} + m} dz, dv = \frac{d^2 n(z)}{dz^2} dz, v = \frac{dn(z)}{dz} \)

\[
l_3 = -e^{-rT} \int_{-\infty}^{\frac{ln K - m}{\sigma \sqrt{T}}} \sigma \sqrt{T} e^{z\sigma \sqrt{T} + m} \frac{d^2 n(z)}{dz^2} dz
= -e^{-rT} \sigma \sqrt{T} K \left(\frac{ln K - m}{\sigma \sqrt{T}}\right) n \left(\frac{ln K - m}{\sigma \sqrt{T}}\right) + e^{-rT} \sigma^2 T \int_{-\infty}^{\frac{ln K - m}{\sigma \sqrt{T}}} \left(e^{z\sigma \sqrt{T} + m}\right) \frac{dn(z)}{dz} dz
\]

let: \( u = e^{z\sigma \sqrt{T} + m}, du = \sigma \sqrt{T} e^{z\sigma \sqrt{T} + m} dz, dv = \frac{dn(z)}{dz} dz, v = n(z) \)
\[
\int_{-\infty}^{\infty} \left( e^{z\sigma\sqrt{T} + m} \right) \left( \frac{dn(z)}{dz} \right) dz = e^{z\sigma\sqrt{T} + m} n(z) \left. \right|_{0}^{\frac{\ln K - m}{\sigma\sqrt{T}}} - \int_{-\infty}^{\infty} n(z) \sigma\sqrt{T} e^{z\sigma\sqrt{T} + m} dz
\]

\[= K n \left( \frac{\ln K - m}{\sigma\sqrt{T}} \right) - \sigma\sqrt{T} \int_{-\infty}^{\frac{\ln K - m}{\sigma\sqrt{T}}} e^{z\sigma\sqrt{T} + m} n(z) dz\]

\[I_3 = -e^{-rT} \sigma\sqrt{T} K \left( -\frac{\ln K - m}{\sigma\sqrt{T}} \right) n \left( \frac{\ln K - m}{\sigma\sqrt{T}} \right) e^{-rT} \sigma^2 T \left( K n \left( \frac{\ln K - m}{\sigma\sqrt{T}} \right) \sigma\sqrt{T} \int_{-\infty}^{\frac{\ln K - m}{\sigma\sqrt{T}}} e^{z\sigma\sqrt{T} + m} n(z) dz \right) \]

We assumed that \( y = z - \sigma\sqrt{T}, dy = dz \).

\[\int_{-\infty}^{\infty} e^{z\sigma\sqrt{T}} n(z) dz = \int_{-\infty}^{\infty} e^{z\sigma\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy\]

\[= \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}x^2} \int_{-\infty}^{\frac{\ln K - m}{\sigma\sqrt{T}}} e^{-\frac{1}{2}(z-\sigma\sqrt{T})^2} dz\]

\[l_3 = -e^{-rT} \sigma\sqrt{T} K \left( -\frac{\ln K - m}{\sigma\sqrt{T}} \right) n \left( \frac{\ln K - m}{\sigma\sqrt{T}} \right) e^{-rT} \sigma^2 T K n \left( \frac{\ln K - m}{\sigma\sqrt{T}} \right) e^{m-rT} \sigma^2 T (\sigma\sqrt{T})^3 \left( \frac{\ln K - m}{\sigma\sqrt{T}} \right) - \sigma\sqrt{T} \]

For \( m = \ln S_0 + \left( r - \frac{1}{2} \sigma^2 \right) T \),

\[d_2 = \frac{\ln K - m}{\sigma\sqrt{T}},\]

\[d_2 = d_1 - \sigma\sqrt{T}, n(-d_2) = n(d_2), \text{and } e^{-rT} K n(d_2) = S_0 n(d_1).\]

We have

\[I_3 = -\sigma\sqrt{T} S_0 n(d_1)(d_1 - \sigma\sqrt{T}) + \sigma^2 T S_0 n(d_1) - e^{\ln S_0 + \left( r - \frac{1}{2} \sigma^2 \right) T - rT + \frac{1}{2} \sigma^2 T (\sigma\sqrt{T})^3} N(-d_1)\]

\[I_3 = \sigma\sqrt{T} S_0 \left( n(d_1) (2\sigma\sqrt{T} - d_1) - \sigma^2 T N(-d_1) \right)\]

Thus,

\[P_{GC-3} = P_{BS} + \mu_3 Q_3\]
With,

\[ \mu_3 = Skewness \]
\[ Q_3 = \frac{S_0 \sigma \sqrt{T} \left( n(d_1)(2\sigma \sqrt{T} - d_1) - \sigma^2 T N(-d_1) \right)}{3!} \]
\[ d_1 = \frac{\ln \frac{S_0 + (r + \sigma^2)^T}{r \sqrt{T}}}{\sigma \sqrt{T}} \]
\[ d_2 = d_1 - \sigma \sqrt{T} \]

Case Studies

In this case study we will calculate the option price of European type of expansion model of Gram-Charlier and Black-Scholes model. After that, the calculations from both models will be compared with the option price in the market. Interest rate risk (r) = 1.25% (Data from the Federal Fund Rate). The stock data observed were Simon Property Group, Inc. (SPG), Citigroup Inc. (C), and American Express Company (AXP). The data taken is daily stock price data for 1 year (Nov 11, 2016 - Nov 11, 2017). Then calculate the Log-return value with the formula:

\[ R_t = \ln \frac{S_T}{S_{T-1}} \]

From Log-return data of the three stocks above, the following data are obtained:

<table>
<thead>
<tr>
<th>STOCK</th>
<th>SPG</th>
<th>C</th>
<th>AXP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock Price ((S_0))</td>
<td>$163.75</td>
<td>$72.25</td>
<td>$93.52</td>
</tr>
<tr>
<td>Average of Return</td>
<td>-0.000425627</td>
<td>0.001184114</td>
<td>0.001799112</td>
</tr>
<tr>
<td>Volatility ((\sigma))</td>
<td>20.65%</td>
<td>18.26%</td>
<td>21.75%</td>
</tr>
<tr>
<td>Risk free rate ((r))</td>
<td>1.25%</td>
<td>1.25%</td>
<td>1.25%</td>
</tr>
<tr>
<td>Time ((T))</td>
<td>0.277777778</td>
<td>0.277777778</td>
<td>0.277777778</td>
</tr>
<tr>
<td>Skewness ((\mu_3))</td>
<td>-0.236470618</td>
<td>0.01097949</td>
<td>7.791851308</td>
</tr>
</tbody>
</table>

After calculation using Microsoft Excel software, the price of put option is as follows:

**Simon Property Group, Inc. Put Price**

Below is the option pricing details for Simon Property Group, Inc.:

<table>
<thead>
<tr>
<th>K</th>
<th>Q3</th>
<th>P-Market</th>
<th>P-BS</th>
<th>P-GC-3</th>
<th>MSE P-BS</th>
<th>MSE P-GC</th>
</tr>
</thead>
<tbody>
<tr>
<td>130,00</td>
<td>-0.20684</td>
<td>7.80</td>
<td>0.088435</td>
<td>0.137347</td>
<td>59.468232</td>
<td>58.716256</td>
</tr>
<tr>
<td>135,00</td>
<td>-0.34603</td>
<td>9.70</td>
<td>0.226925</td>
<td>0.308752</td>
<td>89.739145</td>
<td>88.195534</td>
</tr>
<tr>
<td>140,00</td>
<td>-0.48606</td>
<td>10.01</td>
<td>0.513443</td>
<td>0.628382</td>
<td>90.184600</td>
<td>88.014758</td>
</tr>
<tr>
<td>145,00</td>
<td>-0.57021</td>
<td>13.55</td>
<td>1.040078</td>
<td>1.174915</td>
<td>156.498156</td>
<td>153.142723</td>
</tr>
<tr>
<td>150,00</td>
<td>-0.38118</td>
<td>18.00</td>
<td>3.230104</td>
<td>3.320243</td>
<td>218.149830</td>
<td>215.495275</td>
</tr>
<tr>
<td>160,00</td>
<td>-0.10583</td>
<td>20.70</td>
<td>5.06915</td>
<td>5.094177</td>
<td>244.323461</td>
<td>243.541716</td>
</tr>
<tr>
<td>165,00</td>
<td>0.221035</td>
<td>20.60</td>
<td>7.464904</td>
<td>7.412636</td>
<td>172.530745</td>
<td>173.906573</td>
</tr>
<tr>
<td>180,00</td>
<td>0.845123</td>
<td>28.88</td>
<td>17.72195</td>
<td>17.5221</td>
<td>124.502077</td>
<td>129.001816</td>
</tr>
<tr>
<td>185,00</td>
<td>0.838926</td>
<td>37.20</td>
<td>21.93284</td>
<td>21.73446</td>
<td>233.086142</td>
<td>239.182937</td>
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<tr>
<td>190,00</td>
<td>0.750631</td>
<td>36.93</td>
<td>26.40133</td>
<td>26.22383</td>
<td>110.852805</td>
<td>114.622032</td>
</tr>
<tr>
<td>195,00</td>
<td>0.616546</td>
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<td>31.05433</td>
<td>30.90854</td>
<td>432.459758</td>
<td>438.544823</td>
</tr>
<tr>
<td>K</td>
<td>Q3</td>
<td>P-Market</td>
<td>P-BS</td>
<td>P-GC-3</td>
<td>MSE P-BS</td>
<td>MSE P-GC</td>
</tr>
<tr>
<td>------</td>
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<td>----------</td>
<td>--------</td>
<td>--------</td>
<td>----------</td>
<td>----------</td>
</tr>
<tr>
<td>60.00</td>
<td>-0.11092</td>
<td>3.20</td>
<td>0.058643</td>
<td>0.057425</td>
<td>9.868125</td>
<td>9.875778</td>
</tr>
<tr>
<td>62.50</td>
<td>-0.18303</td>
<td>4.15</td>
<td>0.171469</td>
<td>0.16946</td>
<td>15.828706</td>
<td>15.844700</td>
</tr>
<tr>
<td>65.00</td>
<td>-0.22873</td>
<td>5.10</td>
<td>0.421108</td>
<td>0.418597</td>
<td>21.892030</td>
<td>21.915537</td>
</tr>
<tr>
<td>67.50</td>
<td>-0.20465</td>
<td>5.75</td>
<td>0.890471</td>
<td>0.888224</td>
<td>23.615022</td>
<td>23.636865</td>
</tr>
<tr>
<td>70.00</td>
<td>-0.09725</td>
<td>6.68</td>
<td>1.657794</td>
<td>1.656726</td>
<td>25.222558</td>
<td>25.233283</td>
</tr>
<tr>
<td>72.50</td>
<td>0.06141</td>
<td>7.83</td>
<td>2.772066</td>
<td>2.77274</td>
<td>25.582698</td>
<td>25.575878</td>
</tr>
<tr>
<td>75.00</td>
<td>0.212498</td>
<td>9.15</td>
<td>4.238431</td>
<td>4.240764</td>
<td>24.123507</td>
<td>24.100594</td>
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<tr>
<td>77.50</td>
<td>0.306456</td>
<td>10.30</td>
<td>6.020333</td>
<td>6.023698</td>
<td>18.315551</td>
<td>18.286762</td>
</tr>
<tr>
<td>80.00</td>
<td>0.326593</td>
<td>11.75</td>
<td>8.054886</td>
<td>8.058471</td>
<td>13.653871</td>
<td>13.627383</td>
</tr>
<tr>
<td>82.50</td>
<td>0.287744</td>
<td>12.30</td>
<td>10.27208</td>
<td>10.27524</td>
<td>4.112454</td>
<td>4.099650</td>
</tr>
<tr>
<td>85.00</td>
<td>0.219147</td>
<td>15.42</td>
<td>12.6096</td>
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<td>90.00</td>
<td>0.088025</td>
<td>19.35</td>
<td>17.47113</td>
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<tr>
<td>95.00</td>
<td>0.019539</td>
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<tr>
<td>100.00</td>
<td>-0.00361</td>
<td>28.22</td>
<td>27.40427</td>
<td>27.40423</td>
<td>0.665422</td>
<td>0.665487</td>
</tr>
</tbody>
</table>

Based on the Mean Square Error (MSE) value above, the Gram-Charlier model option price is better than the Black-Scholes model because it has a smaller MSE. In other words, the Gram-Charlier model is closer to the price of the market option than the Black-Scholes model option price.

**Citigroup Inc. Put Price**

Below is the option pricing details for Citigroup Inc.:

<table>
<thead>
<tr>
<th>K</th>
<th>Q3</th>
<th>P-Market</th>
<th>P-BS</th>
<th>P-GC-3</th>
<th>MSE P-BS</th>
<th>MSE P-GC</th>
</tr>
</thead>
<tbody>
<tr>
<td>210.00</td>
<td>0.220275</td>
<td>44.50</td>
<td>45.60635</td>
<td>45.55426</td>
<td>1.224006</td>
<td>1.111463</td>
</tr>
<tr>
<td>220.00</td>
<td>0.071928</td>
<td>52.43</td>
<td>55.51064</td>
<td>55.49363</td>
<td>9.490319</td>
<td>9.385812</td>
</tr>
<tr>
<td>280.00</td>
<td>-0.03513</td>
<td>103.60</td>
<td>115.2795</td>
<td>115.2878</td>
<td>136.409923</td>
<td>136.604040</td>
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</table>

**Mean** 148.494229 149.247554
Based on the Mean Square Error (MSE) value above, the Gram-Charlier model option price is better than the Black-Scholes model because it has a smaller MSE. In other words, the Gram-Charlier model is closer to the price of the market option than the Black-Scholes model option price.

American Express Company Put Price

Below is the option pricing details for American Express Company:

<table>
<thead>
<tr>
<th>K</th>
<th>Q3</th>
<th>P-Pasar</th>
<th>P-BS</th>
<th>P-GC</th>
<th>MSE P-BS</th>
<th>MSE P-GC</th>
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<tr>
<td>72.50</td>
<td>-0.103327267</td>
<td>2.12</td>
<td>0.039645151</td>
<td>-0.765465546</td>
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<td>75.00</td>
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<td>0.091372201</td>
<td>-1.21829319</td>
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<tr>
<td>77.50</td>
<td>-0.241223451</td>
<td>2.65</td>
<td>0.1914317</td>
<td>-1.688145561</td>
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<tr>
<td>82.50</td>
<td>-0.340471886</td>
<td>4.35</td>
<td>0.656427768</td>
<td>-1.996478541</td>
<td>13.642476</td>
<td>40.277790</td>
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<tr>
<td>85.00</td>
<td>-0.326795837</td>
<td>4.40</td>
<td>1.093153054</td>
<td>-1.453191517</td>
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<tr>
<td>87.50</td>
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<td>5.30</td>
<td>1.713527396</td>
<td>-0.282321608</td>
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<tr>
<td>90.00</td>
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<tr>
<td>97.50</td>
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<tr>
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<td>27.40887659</td>
<td>40.537099</td>
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</tr>
</tbody>
</table>

| Mean         | 33.327760 | 27.623784 |
Based on the Mean Square Error (MSE) value above, the option price of the Black-Scholes model is better than the Gram-Charlier model option price because it has a smaller MSE. In other words, the Black-Scholes model is closer to the price of a market option than the Gram-Charlier model option price.

**Conclusion**

Based on the results of the discussion and case studies, the following conclusions can be obtained:

1. The pricing of European type of put options with Gram-Charlier expansion is obtained by a hermite polynomial approach. Put option price obtained is the Black-Scholes put option price plus the equation relating to abnormal skewness and kurtosis.
2. The option price with Gram-Charlier expansion is influenced by several factors: initial stock price ($S_0$), strike price (K), interest rate risk (r), maturity (T), volatility (σ), skewness and kurtosis.
3. Return of stocks with skewness and kurtosis values near 0 and 3, will result in the price of the Gram-Charlier Expansion option approaching the Black-Scholes model option price.
4. In SPG and C put price the Gram-Charlier model option price is much better than the option price with the Black-Scholes model because it has a smaller MSE. In other words, the Gram-Charlier model for this stock is closer to the price of the market option compared to the Black-Scholes model option price, but not for AXP stock.

**References**


https://en.wikipedia.org/wiki/Federal_funds_rate